"Hit and Run" for the efficient generation of weights

Gert van Valkenhoef, Tommi Tervonen,
Nalan Baştürk, Douwe Postmus

EURO 2012, Vilnius, Lithuania, July 2012

Background

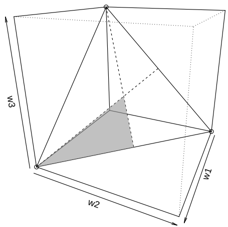
- Many MCDA models consist of per-criterion attractiveness measurement followed by their additive aggregation to an overall measurement of performance, value, or utility.

$$f(f_1(x_1^i), \ldots, f_n(x_n^i)) = \sum_{j=1}^{n} w_j f_j(x_j^i)$$

$$g(g_1(x_1^i, x_1^k), \ldots, g_n(x_n^i, x_n^k)) = \sum_{j=1}^{n} w_j g_j(x_j^i, x_j^k)$$

Simulation-based MCDA

- Weights sum to unity and are non-negative: $(n-1)$-simplex W_n
- In simulation-based MCDA the weights can be imprecise
- Expressed as linear constraints
- Uniform distribution within the feasible weight space $W' \subset W_n$



$w1 > w2 > w3$

Weight constraints

- Ordinal: poor information, but compatible with all models
- Upper- and lower bounds ($0.4 \leq w_1 \leq 0.6$): correspond to meaning of weights in outranking methods
- Intervals for weight ratios ($0.6 \leq (w_2/w_4) \leq 0.7$): correspond to meaning of weights in MAVT/MAUT

- Efficient algorithms exist only for unrestricted, ordinal and lower-bounded weight generation (see Tervonen & Lahdelma, EJOR, 2007)
- But not for arbitrary (combinations of) constraints!

Hit-and-Run

Purpose:

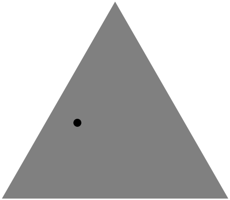
- Uniformly sample over convex volume X

Need:

- Starting point x_0 within X
- Intersect line with boundary of X

Procedure:

- Sample random direction, obtain a line segment through x_i
- Uniformly sample a point within the segment $\rightarrow x_{i+1}$



Hit-and-Run

Purpose:

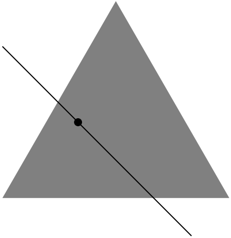
- Uniformly sample over convex volume X

Need:

- Starting point x_0 within X
- Intersect line with boundary of X

Procedure:

- Sample random direction, obtain a line segment through x_i
- Uniformly sample a point within the segment $\rightarrow x_{i+1}$



Hit-and-Run

Purpose:

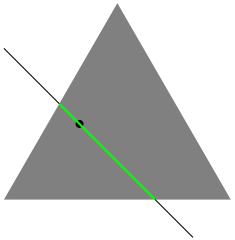
- Uniformly sample over convex volume X

Need:

- Starting point x_0 within X
- Intersect line with boundary of X

Procedure:

- Sample random direction, obtain a line segment through x_i
- Uniformly sample a point within the segment $\rightarrow x_{i+1}$



Hit-and-Run

Purpose:

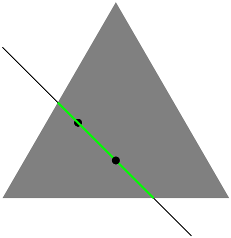
- Uniformly sample over convex volume X

Need:

- Starting point x_0 within X
- Intersect line with boundary of X

Procedure:

- Sample random direction, obtain a line segment through x_i
- Uniformly sample a point within the segment $\rightarrow x_{i+1}$



Hit-and-Run

Purpose:

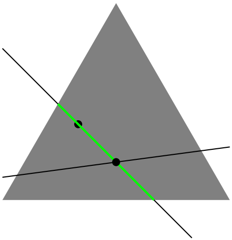
- Uniformly sample over convex volume X

Need:

- Starting point x_0 within X
- Intersect line with boundary of X

Procedure:

- Sample random direction, obtain a line segment through x_i
- Uniformly sample a point within the segment $\rightarrow x_{i+1}$

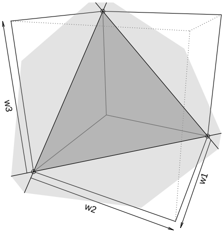


Problems and our solution

1. $Vol(W') = 0 \Rightarrow$ probability of hitting W' also 0
   - Transform the weight space
2. MCMC samplers might get "stuck" in some areas, causing slower convergence to uniformity
   - Assess required thinning

Transformation: idea

- The $(n-1)$-simplex W_n is coincident with the hyperplane $W_n^* = \left\{ w \in R^n : \sum_{j=1}^{n} w_j = 1 \right\}$
- We transform the simplex for sampling in $n-1$ dimensions



Transformation

- The centroid of W_n is at $(1/n, \ldots, 1/n)^T$, so if we translate the plane W_n^* by $(-1/n, \ldots, -1/n)^T$, it forms an $n-1$ dimensional subspace $V \subset R^n$.
- We obtain an orthonormal basis $\{v^1, \ldots, v^{n-1}\}$ of V by first defining a basis of V and then performing orthogonalization and normalization.

Transformation

- To map an arbitrary point $x \in R^{n-1}$ to the target space $w \in W_n^*$, apply an affine transformation: a change of basis followed by a translation.
- Use homogeneous coordinate representation $x = (x_1, x_2, \ldots, x_{n-1}, 1)^T$:

$$w = TBx$$

where
  - B: change-of-basis matrix
  - T: translation matrix
- Both transformations are isometric (preserve uniformity)

Constraints

- Linear constraints defining $W' \subseteq W_n$ need to be defined in $n-1$ dimensions
- Constraint set defining W' is:

$$Cw \leq b \;\; ; \;\; \sum_{i=1}^{n} w_i = 1$$

- Since we sample directly from the plane W_n^*, the equality constraint can be dropped.
- Then the constraints can be expressed in $n-1$ dimensions as:

$$Ax \leq b \;\; ; \;\; A = CTB$$

since $Ax = C(TBx) = Cw$.

Transformation: visualized

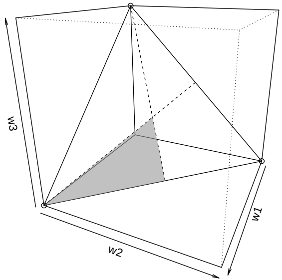


Figure: Constraints in 3D: $Cw \leq b$

Transformation: visualized

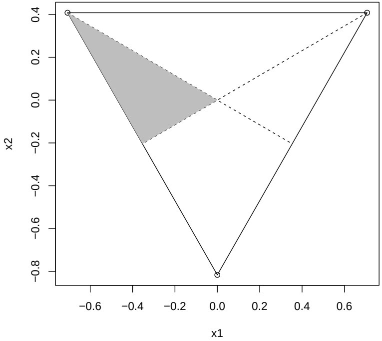


Figure: Constraints in 2D: $Ax = CTBx \leq b$

Transformation: visualized

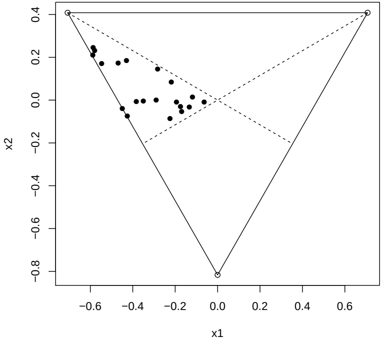


Figure: Hit-and-Run obtains samples x in 2D

Transformation: visualized

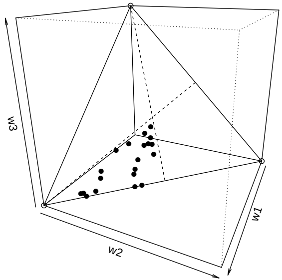


Figure: Transform samples to 3D: $w = TBx$

Line intersection, starting point

- The polytope is defined as $Ax \leq b$, so line intersection is some simple linear algebra
- A starting point can be found with convex combination of points of the polytope (obtainable with LPs or Fukuda-Avis vertex enumeration)

Thinning: computational tests

- HAR mixes with $O^*(n^3)$ iterations, $n = 2 \Rightarrow$ thinning $= 1$

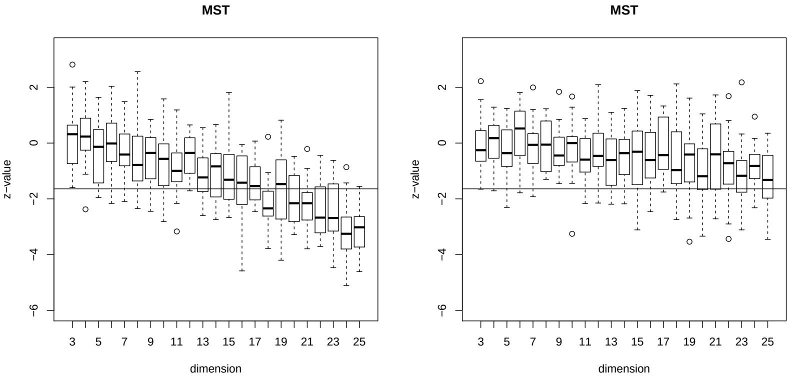
$$f_a(n) = a(n-1)^3 + (1-a)$$

- How much thinning is required?
- How to assess sample uniformity?
  - Minimum Spanning Tree (MST) test (Friedman-Rafsky two-sample test)
  - Coefficient of Variation (COV) of the nearest neighbour-distances
  - Standardized Component-wise Error (SCE)
  - Autocorrelation at lag 25 (decided after visual inspection on exploratory tests)

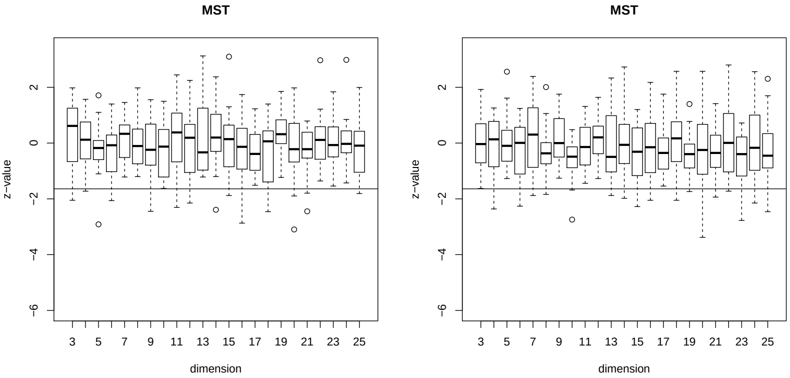
Test setup

- Use ordinal weight information ($w_1 > w_2 \cdots > w_n$)
- Sample Y =10k weight vectors with HAR
- Sample X =10k weight vectors with an efficient procedure
- $n \in \{3, \ldots, 25\}$
- $f_a(n)$, $a \in \{0.125, 0.25, 0.5, 0.75, 1.0\}$
- For each test instance, 20 runs

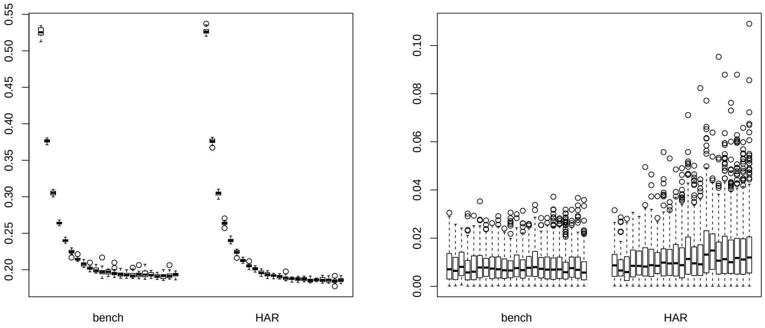
MST metric, HAR, $a = 0.25$ and $a = 0.5$



MST metric, benchmark and HAR with $a = 1.0$

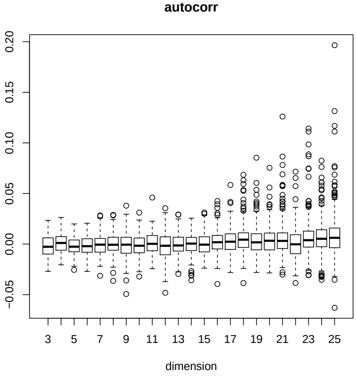


COV and SCE metrics, benchmark and HAR with $f_{1.0}$



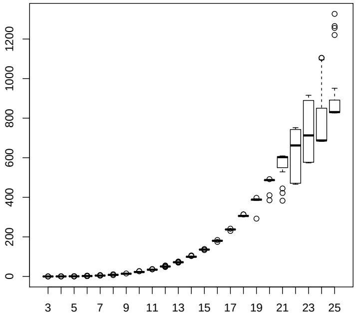
- COV reaches acceptable levels too early (before MST), and depends on dimension $\Rightarrow$ not suitable
- SCE more suitable, but requires enumerating the vertices

Autocorrelation (lag 25) metric, HAR with $f_{1.0}$



- Quite suitable, fast to compute

Results: execution times (s) with thinning $f_{1.0}$



- Up to $n = 15$ within ≈ 3 minutes ($O^*(n^3)$)
- Rejection sampling takes hours for $n > 10$ (exponential)

Conclusions

- Transformation: enables efficient uniform sampling of linearly constrained weight space
- Evaluated four convergence metrics
  - 'Gold standard' MST only usable when HAR not needed
  - Autocorrelation at lag 25 is a good proxy
- The technique is sufficiently fast to be applied in interactive decision analysis with problems of modest sizes
- Sampling code available as the R package 'hitandrun'

Thank you!

Questions?