Efficient weight generation for simulation-based multiple criteria decision analysis

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Background

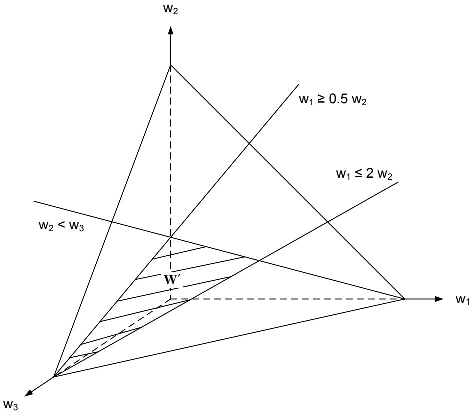
- Many MCDA models consist of per-criterion attractiveness measurement followed by their additive aggregation to an overall measurement of performance, value, or utility.

$$f(f_1(x_1^i), \ldots, f_n(x_n^i)) = \sum_{j=1}^{n} w_j f_j(x_j^i)$$

$$g(g_1(x_1^i, x_1^k), \ldots, g_n(x_n^i, x_n^k)) = \sum_{j=1}^{n} w_j g_j(x_j^i, x_j^k)$$

Simulation-based MCDA

- Weights sum to unity
- In simulation-based MCDA the weights w_j can be imprecise
- Uniform distribution within the feasible weight space W', that is an $(n-1)$-simplex W restricted with the weight constraints



Weight constraints

- Ordinal: poor information, but compatible with all models
- Upper- and lower bounds ($0.4 \leq w_1 \leq 0.6$): correspond to meaning of weights in outranking methods
- Intervals for weight ratios ($0.6 \leq (w_2/w_4) \leq 0.7$): correspond to meaning of weights in MAVT/MAUT

- Efficient algorithms exist only for unrestricted, ordinal and lower-bounded weight generation (see Tervonen & Lahdelma, EJOR, 2007)

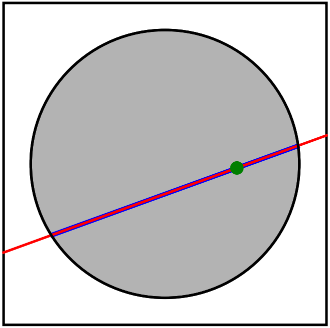
Markov Chain Monte Carlo with Hit-and-Run

Need:

- Starting point within W'
- Bounding box around the polytope to sample in

Procedure:

- Sample random direction, obtain a line segment
- Sample a point within the segment
- If within W', keep as new point, otherwise keep current point as new point

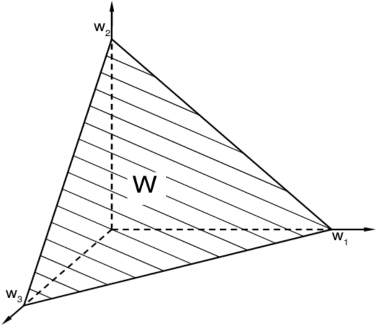


Problems and our solution

1. $Vol(W') \approx 0 \Rightarrow p_{hit}(W') \approx 0$ (transform)
2. MCMC samplers might get “stuck” in some areas, causing slower convergence to uniformity (assess required thinning)

Transformation: idea

- The $(n-1)$-simplex W_n is coincident with the hyperplane $W_n^* = \left\{ w \in R^n : \sum_{j=1}^{n} w_j = 1 \right\}$
- We transform the simplex for sampling in $n-1$ dimensions



Transformation

- The centroid of W_n is at $(1/n, \ldots, 1/n)^T$, so if we translate the plane W_n^* by $(-1/n, \ldots, -1/n)^T$, it forms an $n-1$ dimensional subspace $V \subset R^n$.
- We obtain an orthonormal basis $\{v^1, \ldots, v^{n-1}\}$ of V by first defining a basis of V and then performing orthogonalization and normalization.
- A basis can be defined by choosing $n-1$ vectors, so that for the k^{th} vector the n^{th} component is -1, the k^{th} component is 1, and the others are 0.
- E.g. $\{(1, 0, -1)^T, (0, 1, -1)^T\}$

Transformation

- To map an arbitrary point $x \in R^{n-1}$ to the target space $w \in W_n^*$, apply an affine transformation: a change of basis followed by a translation.
- Use homogeneous coordinate representation $x = (x_1, x_2, \ldots, x_{n-1}, 1)^T$:

$$w = TBx$$

where B is the $(n+1) \times n$ augmented change-of-basis matrix and T the $n \times (n+1)$ translation matrix:

$$B = \begin{pmatrix} v_1^1 & \ldots & v_1^{n-1} & 0 \\ \vdots & & \vdots & \vdots \\ v_n^1 & \ldots & v_n^{n-1} & 0 \\ 0 & \ldots & 0 & 1 \end{pmatrix} \quad ; \quad T = \begin{pmatrix} 1 & 0 & \ldots & 0 & 1/n \\ 0 & 1 & \ldots & 0 & 1/n \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & 1/n \end{pmatrix}$$

- Both transformations are isometric

Constraints

- Linear constraints defining $W' \subseteq W_n$ need to be defined in $n-1$ dimensions
- Constraint set defining W_n is:

$$Cw \leq b \;\; ; \;\; C = -1I_n, \; b = (0, 0, \ldots, 0)^T$$

$$\sum_{i=1}^{n} w_i = 1$$

  where I_n is the $n \times n$ identity matrix.
- Since we sample directly from the plane W_n^*, the equality constraint can be dropped.
- The weight constraints can be represented as additional rows in C and b.
- Then the constraints can be expressed in $n-1$ dimensions as:

$$Ax \leq b \;\; ; \;\; A = CTB$$

  since $Ax = C(TBx) = Cw$.

Line intersection, starting point

- Due to being in convex polytope, we can define the bounding box exactly as the polytope itself (rejection rate $= 0$)
- A starting point can be found with convex combination of vertex points (obtainable with e.g. Fukuda-Avis pivoting algorithm or LPs)
- Details omitted for brevity

Thinning: computational tests

- HAR mixes with $O^*(n^3)$ iterations, $n = 2 \Rightarrow$ thinning $= 1$

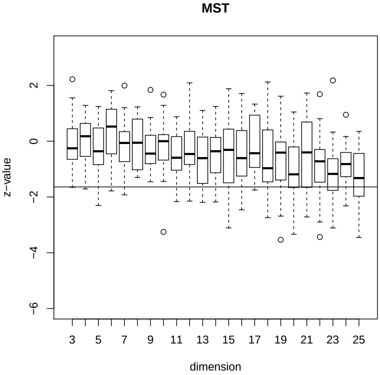
$$f_a(n) = a(n-1)^3 + (1-a)$$

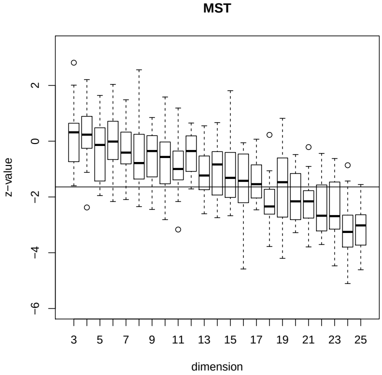
- How much thinning is required?
- How to assess sample uniformity?
  - Minimum Spanning Tree (MST) test (Friedman-Rafsky two-sample test)
  - Coefficient of Variation (COV) of the nearest neighbour-distances
  - Standardized Component-wise Error (SCE)
  - Autocorrelation at lag 25 (decided after visual inspection on exploratory tests)

Test setup

- Use ordinal weight information ($w_1 > w_2 \cdots > w_n$)
- Sample $Y =$10k weight vectors with HAR
- Sample $X =$10k weight vectors with an efficient procedure
- $n \in \{3, \ldots, 25\}$
- $f_a(n)$, $a \in \{0.125, 0.25, 0.5, 0.75, 1.0\}$
- For each test instance, 20 runs

MST metric, HAR, $a = 0.5$ and $a = 0.25$





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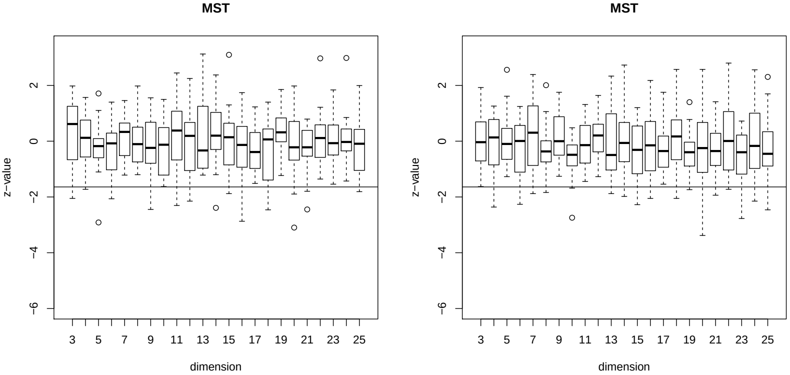
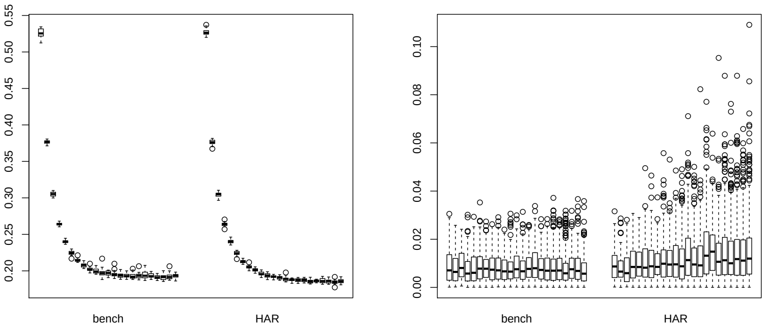
MST metric, benchmark and HAR with $a = 1.0$

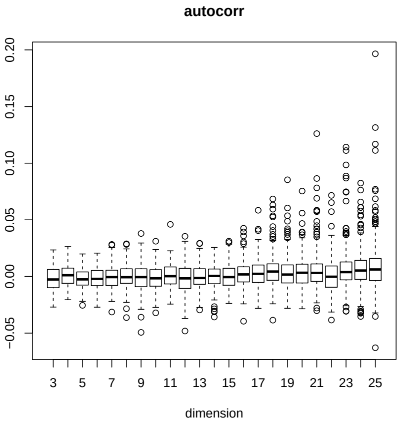


COV and SCE metrics, benchmark and HAR with $f_{1.0}$



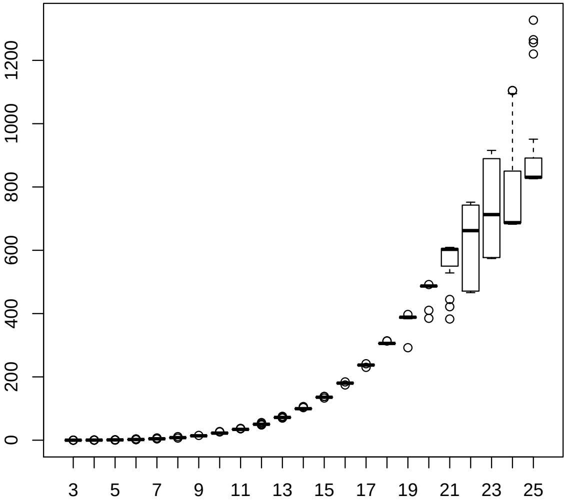
- COV reaches acceptable levels too early (before MST), and depends on dimension $\Rightarrow$ not suitable
- SCE more suitable, but requires enumerating the vertices

Autocorrelation (lag 25) metric, HAR with $f_{1.0}$



- Quite suitable, fast to compute

Results: execution times (s) with thinning $f_{1.0}$



Conclusions

- We provided a transformation technique that enables efficient uniform MCMC sampling within a linearly constrained simplex
- To assess sample uniformity, we evaluated four convergence metrics, of which MST and autocorrelation at lag 25 are the most suitable ones
- The technique is sufficiently fast to be applied in interactive decision analysis with problems of modest sizes
- Sampling code available as the R package 'hitandrun'
- Future research: MCMC sampling for non-convex preference spaces (teaser: to be presented in EURO)

Obrigado pela sua atenção!

Moltes gràcies per la seva atenció!

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