Efficient weight generation for simulation-based multiple criteria decision analysis

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Many MCDA models consist of per-criterion attractiveness measurement followed by their additive aggregation to an overall measurement of performance, value, or utility.

\[
f(f_1(x_1^i), \ldots, f_n(x_n^i)) = \sum_{j=1}^{n} w_j f_j(x_j^i)
\]

\[
g(g_1(x_1^i, x_1^k), \ldots, g_n(x_n^i, x_n^k)) = \sum_{j=1}^{n} w_j g_j(x_j^i, x_j^k)
\]
Simulation-based MCDA

- Weights sum to unity
- In simulation-based MCDA the weights $w_j$ can be imprecise
- Uniform distribution within the feasible weight space $W'$, that is an $(n-1)$-simplex $W$ restricted with the weight constraints
Weight constraints

- Ordinal: poor information, but compatible with all models
- Upper- and lower bounds \((0.4 \leq w_1 \leq 0.6)\): correspond to meaning of weights in outranking methods
- Intervals for weight ratios \((0.6 \leq (w_2/w_4) \leq 0.7)\): correspond to meaning of weights in MAVT/MAUT

- Efficient algorithms exist only for unrestricted, ordinal and lower-bounded weight generation (see Tervonen & Lahdelma, EJOR, 2007)
Markov Chain Monte Carlo with Hit-and-Run

Need:
- Starting point within $W'$
- Bounding box around the polytope to sample in

Procedure:
- Sample random direction, obtain a line segment
- Sample a point within the segment
- If within $W'$, keep as new point, otherwise keep current point as new point
Problems and our solution

1. $\text{Vol}(W') \approx 0 \Rightarrow p_{\text{hit}}(W') \approx 0$ (transform)

2. MCMC samplers might get “stuck” in some areas, causing slower convergence to uniformity (assess required thinning)
The $(n-1)$-simplex $W_n$ is coincident with the hyperplane $W_n^* = \left\{ w \in \mathbb{R}^n : \sum_{j=1}^{n} w_j = 1 \right\}$

We transform the simplex for sampling in $n-1$ dimensions
The centroid of $W_n$ is at $(1/n, \ldots, 1/n)^T$, so if we translate the plane $W_n^*$ by $(-1/n, \ldots, -1/n)^T$, it forms an $n - 1$ dimensional subspace $V \subset \mathbb{R}^n$.

We obtain an orthonormal basis $\{v^1, \ldots, v^{n-1}\}$ of $V$ by first defining a basis of $V$ and then performing orthogonalization and normalization.

A basis can be defined by choosing $n - 1$ vectors, so that for the $k^{th}$ vector the $n^{th}$ component is $-1$, the $k^{th}$ component is $1$, and the others are 0.

E.g. $\{(1, 0, -1)^T, (0, 1, -1)^T\}$
To map an arbitrary point \( x \in \mathbb{R}^{n-1} \) to the target space \( w \in W_n^* \), apply an affine transformation: a change of basis followed by a translation.

Use homogeneous coordinate representation
\[
x = (x_1, x_2, \ldots, x_{n-1}, 1)^T:
\]

\[
w = TBx
\]

where \( B \) is the \((n + 1) \times n\) augmented change-of-basis matrix and \( T \) the \( n \times (n + 1) \) translation matrix:

\[
B = \begin{pmatrix}
v_1^1 & \ldots & v_1^{n-1} & 0 \\
\vdots & \ddots & \vdots & \vdots \\
v_n^1 & \ldots & v_n^{n-1} & 0 \\
0 & \ldots & 0 & 1
\end{pmatrix} \quad ; \quad T = \begin{pmatrix}
1 & 0 & \ldots & 0 & 1/n \\
0 & 1 & \ldots & 0 & 1/n \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & 1 & 1/n
\end{pmatrix}
\]

Both transformations are isometric
Constraints

- Linear constraints defining $W' \subseteq W_n$ need to be defined in $n - 1$ dimensions.
- Constraint set defining $W_n$ is:
  \[ Cw \leq b \ ; \ C = -1I_n, \ b = (0, 0, \ldots, 0)^T \]
  \[ \sum_{i=1}^{n} w_i = 1 \]
  where $I_n$ is the $n \times n$ identity matrix.
- Since we sample directly from the plane $W_n^*$, the equality constraint can be dropped.
- The weight constraints can be represented as additional rows in $C$ and $b$.
- Then the constraints can be expressed in $n - 1$ dimensions as:
  \[ Ax \leq b \ ; \ A = CTB \]
  since $Ax = C(TBx) = Cw$. 
Due to being in convex polytope, we can define the bounding box exactly as the polytope itself (rejection rate = 0)

A starting point can be found with convex combination of vertex points (obtainable with e.g. Fukuda-Avis pivoting algorithm or LPs)

Details omitted for brevity
Thinning: computational tests

- HAR mixes with $O^*(n^3)$ iterations, $n = 2 \implies$ thinning = 1
  
  $$f_a(n) = a(n - 1)^3 + (1 - a)$$

- How much thinning is required?

- How to assess sample uniformity?
  
  - Minimum Spanning Tree (MST) test (Friedman-Rafsky two-sample test)
  - Coefficient of Variation (COV) of the nearest neighbour-distances
  - Standardized Component-wise Error (SCE)
  - Autocorrelation at lag 25 (decided after visual inspection on exploratory tests)
Test setup

- Use ordinal weight information ($w_1 > w_2 \cdots > w_n$)
- Sample $Y = 10k$ weight vectors with HAR
- Sample $X = 10k$ weight vectors with an efficient procedure
- $n \in \{3, \ldots, 25\}$
- $f_a(n), a \in \{0.125, 0.25, 0.5, 0.75, 1.0\}$
- For each test instance, 20 runs
MST metric, HAR, $a = 0.5$ and $a = 0.25$
MST metric, benchmark and HAR with $a = 1.0$
- COV reaches acceptable levels too early (before MST), and depends on dimension $\Rightarrow$ not suitable
- SCE more suitable, but requires enumerating the vertices
Autocorrelation (lag 25) metric, HAR with $f_{1.0}$

- Quite suitable, fast to compute
Results: execution times (s) with thinning $f_{1.0}$
Conclusions

- We provided a transformation technique that enables efficient uniform MCMC sampling within a linearly constrained simplex.
- To assess sample uniformity, we evaluated four convergence metrics, of which MST and autocorrelation at lag 25 are the most suitable ones.
- The technique is sufficiently fast to be applied in interactive decision analysis with problems of modest sizes.
- Sampling code available as the R package 'hitandrun'.
- Future research: MCMC sampling for non-convex preference spaces (teaser: to be presented in EURO).
Obrigado pela sua atenção!

Moltes gràcies per la seva atenció!